5/H–29 (vi) (Syllabus–2015)

2 0 1 7

(October)

MATHEMATICS

(Honours)

(Differential Equations and Advanced Dynamics)

(GHS–52)

Marks : 75

Time : 3 hours

The figures in the margin indicate full marks for the questions

Answer Differential Equations and Advanced Dynamics in separate books

Answer **five** questions, choosing **one** from each Unit

UNIT—I

1. (*a*) Solve the differential equation

$$x^3 \frac{d^3y}{dx^3} + 3x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = x + \log x$$ 6

(b) Solve the equation

$$(1+x+x^2)\frac{d^3y}{dx^3}+(3+6x)\frac{d^2y}{dx^2}+6\frac{dy}{dx}=0$$ 6

(c) Solve the following equation : 3

$$\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y=e^{5x}$$

2. (a) Apply the method of variation of parameters to solve the equation

$$\frac{d^2y}{dx^2}+k^2y=\sec kx$$ 6

(b) Solve the following simultaneous differential equations : 6

$$\frac{dx}{dt}+4x+3y=t$$

$$\frac{dy}{dt}+2x+5y=e^t$$

(c) Solve the following equation : 3

$$\frac{dx}{z(x+y)}=\frac{dy}{z(x-y)}=\frac{dz}{x^2+y^2}$$

UNIT—II

(In this unit, $p=\frac{\partial z}{\partial x},\ q=\frac{\partial z}{\partial y}$)

3. (a) Form a partial differential equation by eliminating a, b, c from

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$ 6

(b) Solve the following equation : 3

$$\frac{y^2zp}{x}+zxq=y^2$$

(c) Apply Charpit's method to find the complete integral of

$$(p^2+q^2)y=qz$$ 6

4. (a) Find the integral surface of the partial differential equation $(x-y)p+(y-x-z)q=z$ through the circle $z=1,\ x^2+y^2=1$. 6

(b) Find the complete integral and singular integral of

$$z=px+qy+c\sqrt{1+p^2+q^2}$$ 6

(c) Find the complete integral of $p^2=zq$. 3

UNIT—III

5. Establish the formula

$$\frac{d^2u}{d\theta^2}+u=\frac{P}{h^2u^2},\ \dot{\theta}=hu^2$$

where $u=\frac{1}{r}$ for the motion of a particle describing a central orbit under an attraction P per unit mass.

If $P=\mu u^5$, find the speed v with which the particle can describe the circle $r=a$.

If the particle moves under this attraction with the same areal constant as the circular path and

$$\dot{r}=-\frac{3v}{4\sqrt{2}},\ \text{where } r=2a,\ \theta=0$$

find the equation of the spiral path of the particle and show that as $\theta\to\infty$, the path is asymptotic to the circle $r=a$. 6+2+7=15

6. *(a)* A particle slides in a verticle plane down a rough cycloidal arc whose axis is vertical and vertex downwards, starting from a point, where the tangent makes an angle θ with the horizon and coming to rest at the vertex. Show that $\mu e^{\mu\theta}=\sin\theta-\mu\cos\theta$, μ being the coefficient of friction. 8

(b) If $P=\mu(u^2-au^3)$, where $a>0$ and a particle is projected from an apse at a distance a from the centre of force with a velocity $\sqrt{\mu c/a^2}$, where $a>c$, then prove that the other apsidal distance of the orbit is $\frac{a(a+c)}{a-c}$. 7

UNIT—IV

7. *(a)* For a coplanar rigid system, prove that the principal moments of inertia at a point are the extreme values of the moments of inertia at that point. Show further that these extreme values are given by

$$I_{\min}=\frac{1}{2}\left[A+B-\sqrt{(B-A)^2+4F^2}\right]$$

$$I_{\max}=\frac{1}{2}\left[A+B+\sqrt{(B-A)^2+4F^2}\right]$$

where A, B and F have their usual meanings. 8

(b) A uniform rigid rod AB moves so that A and B have velocities $\vec{U}_A$ and $\vec{U}_B$ at any instant. Show that the kinetic energy is then

$$T=\frac{1}{6}M\left[\vec{U}_A^2+\vec{U}_A.\vec{U}_B+\vec{U}_B^2\right]$$

where M is the mass of the rod. 7

8. *(a)* A uniform solid rectangular block is of mass M and have dimensions $2a$, $2b$, $2c$. Find the equation of the momental ellipsoid for a corner O of the block, referred to edges through O as coordinate axes. Determine the moment of inertia about OO', where O' is the point diagonally opposite to O. 10

(b) Find the moment of inertia of a rectangular lamina of sides $2a$, $2b$ about a line parallel to one of its sides. 5

UNIT—V

9. *(a)* A uniform rod AB of mass $2m$ is freely jointed at B to a second rod BC of mass m. The rods lie on a smooth horizontal plane at right angles to each other and an impulse I is applied to AB at A in a direction parallel to BC. Find the initial velocity of BC and prove that the kinetic energy generated is $\frac{5}{6} I^2 / m$. 6+2=8

(b) A circular hoop of radius a, rotating in a vertical plane with spin ω and with its centre at rest, is in contact with a rough plane inclined at an angle α, the angle of friction for the surfaces in contact also being α. Show that, if the initial slip velocity is down the plane, the hoop

(Continued)

remains stationary for the time $\frac{a\omega}{g\sin\alpha}$ and the hoop rolls down the plane with acceleration $\frac{1}{2} g\sin\alpha$. 7

10. *(a)* A uniform rod is placed on a horizontal table with two-thirds of its length hanging over the edge of the table. If the rod is at right angles to the edge and is released, show that it will begin to slip when the rod has turned through an angle of

$$\tan^{-1}\left(\frac{1}{2}\mu\right)$$

where μ is the coefficient of friction between rod and table. 7

(b) A uniform circular cylinder of mass M and radius a rolls down a rough inclined plane, inclined at an angle α to the horizontal. Prove that its velocity down the plane is given by

$$v = \frac{2}{3} gt\sin\alpha + v_0$$

and that its angular velocity at time t is given by

$$\omega = \frac{2gt\sin\alpha}{3a} + \omega_0$$

where v_0 and ω_0 are the initial values of v and ω respectively at time $t = 0$. 8

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(GHS–52)

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The figures in the margin indicate full marks for the questions

Write Units I and II together in one answer script and Units III, IV and V together in another answer script

Answer **five** questions, choosing **one** from each Unit

UNIT—I

1. *(a)* Solve : 4

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{2x}\sin x$$

(b) Solve the equation

$$x\frac{d^2y}{dx^2}-\frac{dy}{dx}+4x^3y=x^5$$

by changing the independent variable. 6

(c) Solve : 5

$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}=\frac{12\log x}{x^2}$$

2. (a) Test for exactness and solve

$$(1+x^2)\frac{d^2y}{dx^2}+4x\frac{dy}{dx}+2y=\sec^2 x$$

given that $y=0$, $\frac{dy}{dx}=1$ when $x=0$. 6

(b) Solve the simultaneous equation : 4

$$\frac{dx}{mz-ny}=\frac{dy}{nx-lz}=\frac{dz}{ly-mx}$$

(c) Show that the equation

$$(yz+xyz)dx+(zx+xyz)dy+(xy+xyz)dz=0$$

is integrable and find its solution. 5

UNIT—II

$\left(\text{In this Unit, } p=\frac{\partial z}{\partial x}, q=\frac{\partial z}{\partial y}\right)$

3. (a) Form a partial differential equation by eliminating the arbitrary function ϕ from $\phi(x+y+z, x^2+y^2-z^2)=0$. What is the order of this partial differential equation? 6

(b) Solve : 4

$$x(y^2+z)p-y(x^2+z)q=z(x^2-y^2)$$

(c) Find complete and singular integrals by Charpit's method of

$$2xz-px^2-2qxy+pq=0$$ 3+2

4. (a) Find the equation of the integral surface satisfying the differential equation $4yzp+q+2y=0$ and passing through $y^2+z^2=1$, $x+z=2$. 6

(b) Prove that the complete integral of $z=px+qy-2p-3q$ represents all possible planes through the point (2, 3, 0). Also find the envelope of all planes represented by the complete integral. 6

(c) Find the complete integral of $p^2+q^2=m^2$ where m is a constant. 3

UNIT—III

5. (a) A particle slides in a vertical plane down a rough cycloidal arc whose axis is vertical and vertex downwards, starting from a point, where the tangent makes

an angle θ with the horizon and coming to rest at the vertex. If μ be the coefficient of friction, show that

$$\mu(\sec\theta \cdot e^{\mu\theta} + 1) = \tan\theta$$ 8

(b) A particle moves under a central force $\mu(3a^3u^4 + 8au^2)$ and is projected from an apse at a distance a from the centre of force with velocity $\sqrt{10\mu}$. Show that the second apsidal distance is half of the first. 7

6. *(a)* A particle is describing an ellipse of eccentricity e about a centre of force at a focus. Prove that—

(i) $rav^2 = \mu(2a - r)$

(ii) $h^2 + \mu ae^2 = \mu a$

with the usual notations. 3+2

(b) If a particle P slides down a rough cycloid whose axis is vertical and vertex lowest, then show that

$$v^2 = \frac{4ag}{1+\mu^2}[A^2e^{2\mu\theta} - (\sin\theta - \mu\cos\theta)^2]$$

where μ is the coefficient of friction, θ the angle made by the tangent at P and A being any constant depending on initial conditions. 10

UNIT—IV

7. *(a)* Define the term 'equimomental systems'. State and prove the necessary and sufficient conditions for two systems to be equimomental. 2+8

(b) Find an equimomental system of particles for a uniform rod AB of mass M. 5

8. *(a)* Show that a uniform solid cuboid of mass M is equimomental with—

(i) masses $\frac{1}{24}M$ at the midpoints of its edges and $\frac{1}{2}M$ at its centre;

(ii) masses $\frac{1}{24}M$ at its corners and $\frac{2}{3}M$ at its centre. 5+5

(b) A square of side a has particles of masses m, $2m$, $3m$ and $4m$ at its vertices. Show that the principal moments of inertia at the centre of the square are $2ma^2$, $3ma^2$, $5ma^2$ and find the directions of the principal axes. 5

UNIT—V

9. *(a)* A uniform rod AB of length $2a$ can turn freely about one end A, and at time $t = 0$ is hanging from A in equilibrium under gravity. The end A is then set in motion in a horizontal straight line so that when $t \geq 0$, $OA = vt + \frac{1}{2}ft^2$, where O is a fixed point in the line and v, f are constants. Show that the initial angular velocity of the rod is $\frac{3v}{4a}$ and it will make complete revolutions about A, if

$$3v^2 > 8a[g + (g^2 + f^2)^{1/2}]$$ 9

(b) A uniform circular disc of mass M and radius a is rotating in its plane with initial angular velocity ω, its centre being at rest. If a point on the rim be suddenly fixed, find the new angular velocity of the disc and the velocity of its centre. 6

10. *(a)* A uniform rod AB of mass M and length $2a$ lies at rest on a smooth horizontal table. An impulse J is applied at A in the plane of the table and perpendicular to the rod. Determine the velocity of the centroid and the angular velocity of the rod. 7

(b) A uniform rod of mass m and length $2a$ falls from rest in a vertical position with one end fixed on a table which is so rough that slipping never occurs. Show that when the rod is inclined to the vertical at an angle θ, the angular velocity is $\sqrt{\frac{3g}{a}} \sin\left(\frac{\theta}{2}\right)$ and that the rod will leave the table when the inclination to the vertical is $\cos^{-1}\left(\frac{1}{3}\right)$. 8

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5/H–29 (vi) (Syllabus–2015)

2019

(October)

MATHEMATICS

(Honours)

(GHS–52)

(Differential Equations and Advanced Dynamics)

Marks : 75

Time : 3 hours

The figures in the margin indicate full marks for the questions

Write Units I and II together in one answer script and Units III, IV and V together in another answer script

Answer **five** questions, choosing **one** from each Unit

UNIT—I

1. *(a)* Solve :

$$\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (4x^2 - 3)y = e^{x^2}$$ 6

(b) Solve :

$$\frac{d^2y}{dx^2}-\frac{2}{x}\left(\frac{dy}{dx}\right)+\left(a^2+\frac{2}{x^2}\right)y=0$$

by changing to normal form. 6

(c) Solve :

$$(y+z)dx+(z+x)dy+(x+y)dz=0$$ 3

2. (a) By the method of variation of parameters, solve the equation

$$\frac{d^2y}{dx^2}+4y=4\tan 2x$$ 6

(b) Solve the simultaneous differential equations

$$\frac{dx}{dt}+5x+y=e^t$$

$$\frac{dy}{dt}-x+3y=e^{2t}$$ 6

(c) Solve :

$$\frac{dx}{x^2-yz}=\frac{dy}{y^2-zx}=\frac{dz}{z^2-xy}$$ 3

UNIT—II

$$\left(\text{In this unit, } p=\frac{\partial z}{\partial x},\ q=\frac{\partial z}{\partial y}\right)$$

3. (a) Form a partial differential equation by eliminating the arbitrary functon ϕ from $lx+my+nz=\phi(x^2+y^2+z^2)$. What is the order of this partial differential equation? 6

(b) Solve :

$$(y^2+z^2-x^2)p-2xyq+2zx=0$$ 5

(c) Solve :

$$p^2+q^2=1$$ 4

4. (a) Apply Charpit's method to find complete integral of

$$z^2(p^2z^2+q^2)=1$$ 5

(b) Find the complete integral and singular integral of

$$z=px+qy+p^2+q^2$$ 6

(c) Find the complete integral of

$$\sqrt{p}+\sqrt{q}=2x$$ 4

UNIT—III

5. *(a)* A particle moves under a central repulsive force $m\mu/(\text{distance})^3$ and is projected from an apse at a distance a with velocity V. Show that the equation to the path is $r\cos k\theta = a$ and that the angle θ described in time t is $\frac{1}{k}\tan^{-1}\left(\frac{kV}{a}t\right)$ where $k^2 = \frac{\mu + a^2V^2}{a^2V^2}$. 4+4=8

(b) A heavy particle slides down a rough cycloid whose base is horizontal and vertex downwards. Show that if it starts from rest at the cusp and comes to rest at the vertex, then $\mu^2 = e^{-\mu\pi}$. 7

6. *(a)* A particle P, of unit mass, moves under the action of a force of magnitude $\frac{\mu}{SP^2}$, directed towards a fixed point S. If the velocity of P is V when SP is r, then show that the path of P is a conic having S as focus and that the conic is an ellipse, parabola or hyperbola according as V^2 is less than, equal to or greater than $\frac{2\mu}{r}$. 4+3=7

(b) A particle starts from rest from the cusp of a rough cycloid whose axis is

vertical and vertex downwards. Show that its velocity at the vertex to its velocity at the same point when the cycloid is smooth is

$$(e^{-\mu\pi} - \mu^2)^{\frac{1}{2}} : (1+\mu^2)^{\frac{1}{2}}$$ 8

UNIT—IV

7. *(a)* Prove that the moment of inertia of a triangular lamina about any line in its plane is given by

$$I = \frac{1}{6}M[h_1^2 + h_2^2 + h_3^2 + h_1h_2 + h_2h_3 + h_3h_1]$$

where M is the mass of the lamina and h_1, h_2, h_3 are respectively the distances of the vertices of the triangle from the line. 8

(b) A uniform solid rectangular block is of mass M and have dimensions $2a$, $2b$, $2c$. Find the equation of the momental ellipsoid for a corner O of the block, referred to edges through O as co-ordinates axes. 7

8. *(a)* The lengths AB and AD of the sides of a rectangle $ABCD$ are $2a$, $2b$; show that the inclination of AB and one of the principal axis at A is

$$\frac{1}{2}\tan^{-1}\frac{3ab}{2(a^2 - b^2)}$$ 5

(b) Find the moment of inertia of the uniform triangular lamina *ABC*, of mass *M*, about the side *BC*. 5

(c) Prove that the momental ellipsoid, at the centre of the elliptic plate is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\left(\frac{1}{a^2}+\frac{1}{b^2}\right)z^2 = \text{constant}$$ 5

UNIT—V

9. *(a)* A perfectly rough circular hoop of radius a rolls on a horizontal floor with velocity U towards an inelastic step of height $h\left(<\frac{a}{2}\right)$, the plane of the hoop being vertical and perpendicular to the edge of the step. Prove that the hoop can mount the step without losing contact at any stage if

$$4a^2hg < U^2(2a-h)^2 < 4a^2(a-h)g$$ 7

(b) A uniform heavy solid hemisphere of radius a is held at rest with its base vertical and its curved surface in contact with a horizontal plane. If the hemisphere is released when the plane is rough enough to prevent slipping, then show that the angle θ that the base makes with the horizontal at time t is such that

$$\left(\frac{d\theta}{dt}\right)^2 = \frac{15g\cos\theta}{a(28-15\cos\theta)}$$ 8

10. *(a)* A uniform rod is placed with one end in contact with a horizontal table and is then at inclination α to the horizontal and is allowed to fall. When it becomes horizontal, show that its angular velocity is $\sqrt{3g\sin\alpha/2a}$, if the table is perfectly rough, show that the rod never leaves the table. 3+3=6

(b) A rigid body of mass *M* rotates about a horizontal axis through a point *O* in it. Show that the motion is simple harmonic. Find the period and the length of the equivalent simple pendulum. Show further that the minimum value of the length of the equivalent simple pendulum is $2k$, where k is the radius of gyration of the rigid body. 3+3+3=9

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2017

(October)

MATHEMATICS

(Honours)

(Elementary Number Theory and Advanced Algebra)

(GHS–51)

Marks : 75

Time : 3 hours

The figures in the margin indicate full marks for the questions

Answer Elementary Number Theory and Advanced Algebra in two separate books

Answer **five** questions, choosing **one** from each Unit

UNIT—I

(Elementary Number Theory)

1. *(a)* State whether the following statements are True or False with brief justification (a, b, c, n denote integers) (any *five*) : 2×5=10

(i) $4|(n^2+2)$ for some integer n.

(ii) If $(a, b)=1$ and $c|a$ then $(b, c)=1$.

(iii) If $b \mid a^2 + 1$, then $b \mid a^4 + 1$.

(iv) The only prime of the form $n^3 - 1$ is 7.

(v) If $(n, 7) = 1$ then $7 \mid n^6 - 1$.

(vi) $n^2 + n + 41$ is prime for every positive integer n.

(b) Prove that there is an infinite number of primes. 5

2. *(a)* State and prove Fermat's Little theorem. 1+5=6

(b) What is the last digit in the decimal representation of 3^{100}? 4

(c) Prove that $a^5 \equiv a \pmod{10}$ for every integer a. 5

UNIT—II

3. *(a)* State and prove Chinese Remainder theorem. 1+4=5

(b) Solve the following system of linear congruences : 5

$$x \equiv 2 \pmod 3$$
$$x \equiv 3 \pmod 5$$
$$x \equiv 2 \pmod 7$$

(c) Prove that $\phi(5n) = 5\phi(n)$ if and only if 5 divides n. 5

4. *(a)* Find the highest power of 2 which divides 533!. 2

(b) Evaluate : 2

$$\sum_{j=1}^{\infty} \mu(j\,!)$$

(c) Evaluate $\sigma(n)$ and $\tau(n)$ for $n = 3000$. 2+2=4

(d) Evaluate $\phi(3125)$. 2

(e) For any real numbers x and y, prove that $[x + y] \le [x] + [y] + 1$. 5

UNIT—III

(Advanced Algebra)

5. *(a)* Prove that the intersection of any two normal subgroups of a group G is a normal subgroup of G. 5

(b) If f is a homomorphism of group G into a group G' with kernel K, then prove that K is a normal subgroup of G. 5

(c) Prove that any finite integral domain is a field. 5

6. *(a)* If R is the additive group of real numbers and R^+ is the multiplicative group of positive real numbers, then show that the mapping $f: R^+ \to R$ such that $f(x) = \log x$; $x \in R^+$ is an isomorphism. 5

(b) If R is a ring such that $a^2 = a$, for all $a \in R$, prove that $a + a = 0$, for all $a \in R$. 2

(c) The set M of 2×2 matrices over the field of real numbers is a ring with respect to matrix addition and multiplication. Does this ring possess zero divisors? Justify your answer. 2

(d) If R is a commutative ring and $a \in R$, then prove that the set $Ra = \{ra : r \in R\}$ is an ideal of R. 3

(e) Define maximal ideal of a ring R. Is $\{0\}$ in the ring of integers $\mathbb{Z}$ a maximal ideal? Justify your answer. 2+1=3

UNIT—IV

7. *(a)* Prove that a field has only two ideals 0 and itself. 3

(b) Determine all the ideals in $\mathbb{Z}_6$. 3

(c) Define units. Determine the number of units in the ring of integers. 2+2=4

(d) Consider the ring $\mathbb{Z}$. In this ring $5\mathbb{Z} = \{5k : k \in \mathbb{Z}\}$ is an ideal of $\mathbb{Z}$. How many distinct cosets are there in the quotient ring $\mathbb{Z}/5\mathbb{Z}$? Is this quotient ring a field? Justify your answer. 2+3=5

8. *(a)* Let $\mathbb{R}$ be an integral domain and $a, b \in \mathbb{R}$. When do we say the following?

(i) a and b are associates in $\mathbb{R}$

(ii) a is an irreducible element in $\mathbb{R}$

(iii) a is a prime element in $\mathbb{R}$ 2×3=6

(b) *(i)* Show that the polynomial $x^2 + x + 4$ is irreducible over F, the field of integers modulo 11. 3

(ii) Prove that 3 is not a prime element in $\mathbb{Z}[\sqrt{-5}]$. 4

(iii) Find an associate of a non-zero element in $\mathbb{Z}$. 2

UNIT—V

9. *(a)* Let $V(F)$ be a vector space over a field F and let $\{\alpha_1, \alpha_2, ..., \alpha_n\}$ be a set of vectors of V. When is this set of vector said to be linearly independent? Give an example of a finite set of vectors which is linearly independent in the vector space $\mathbb{R}^3$ and $\mathbb{R}$. 2+2=4

(b) Show that the subset {(1, 2, 0), (0, 3, 1) (–1, 0, 1)} of the vector space $V_3(R)$, where R is the field of real numbers, is linearly independent. 5

(c) Prove that each subspace W of a finite dimensional vector space $V(F)$, Fa, field of dimension n is a finite dimensional space with dim $m \leq n$. 6

10. *(a)* Let T be the linear operator on $\mathbb{R}^3$ defined by

$$T(x_1, x_2, x_3) = (3x_1 + x_3, -2x_1 + x_2, -x_1 + 2x_2 + 4x_3)$$

What is the matrix of T with respect to the basis $\beta = \{(1,0,0), (0,1,0), (0,0,1)\}$? Show that T is invertible. 3+2=5

(b) Let $T: R^3 \to R^3$ be a linear transformation defined by

$$T(a_1, a_2, a_3) = (a_1, -a_2, 2a_3).$$

Find the null space of T and range T. 6

(c) Let $V = \mathbb{R}^4$ the real vector space and let $S = \{(2,0,0,1), (-1,0,1,0)\}$. Find $L(S)$ i.e., the set of all linear combinations of finite sets of elements of S. 4

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2018

(October)

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(Honours)

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(GHS–51)

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UNIT—I

(Elementary Number Theory)

1. *(a)* State whether the following statements are True or False with brief justification (a, b, c, n denote integers) (any *five*) : 2×5=10

(i) If $a|c$ and $b|c$, then $ab|c$.

(ii) If $5\nmid(n-1)$, $5\nmid n$ and $5\nmid(n+1)$, then $5|n^2+1$.

(iii) $(a, bc) = 1 \Rightarrow (a, b) = 1$ and $(a, c) = 1$.

(iv) If $(a, b) = 1$, then $(a^2, b^2) = 1$.

(v) If $(a, b) = (a, c)$, then $[a, b] = [a, c]$.

(vi) $4 \nmid a^2 - 2$ for any integer a.

(b) Prove that if $a|n$, then $2^a - 1 \mid 2^n - 1$. 3

(c) Find the remainder, when the sum $S = 1! + 2! + 3! + \cdots + 1000!$ is divided by 8. 2

2. (a) State and prove Wilson's theorem. 1+5=6

(b) Prove that $n^5 - n$ is divisible by 30; for every integer n. 4

(c) Find the remainder, when 3^{247} is divided by 17. 3

(d) Find the number of positive integers less than 3600 that are relatively prime to 3600. 2

UNIT—II

3. (a) Solve the linear congruence $12x \equiv 44 \pmod{59}$. 3

(b) Solve the following systems of linear congruence : 5

$$x \equiv 3 \pmod{11}$$
$$x \equiv 5 \pmod{19}$$
$$x \equiv 10 \pmod{29}$$

(c) For any real number x, prove that
$$[x] + [-x] = \begin{cases} 0, & \text{if } x \text{ is an integer} \\ -1, & \text{otherwise} \end{cases}$$ 4

(d) If n is an odd integer, prove that $\phi(2n) = \phi(n)$. 3

4. (a) Define Möbius function $\mu(n)$. 1

(b) Evaluate : 2
$$\sum_{j=1}^{\infty} \mu(j!)$$

(c) Prove that 4
$$\prod_{d|n} d = n^{\frac{\tau(d)}{2}}$$

(d) Define the arithmetic function $\tau(n)$ for positive integers n and show that it is a multiplicative function. 4

(e) Evaluate $\sigma(4752)$ and $\tau(4752)$. 4

UNIT—III

(Advanced Algebra)

5. (a) If G is a group and H a subgroup of index 2 in G, prove that H is a normal subgroup of G. 5

(b) Prove that the necessary and sufficient condition for a homomorphism f of a group G into a group G' with Kernel K to be an isomorphism of G into G' is that $K = \{e\}$, where e is the identity of G. 5

(c) Prove that every field is an integral domain. 5

6. *(a)* Show that the set $Z[i]$ of Gaussian integers (i.e. the set of complex numbers $a + ib$, where a and b are integers) forms a ring under ordinary addition and multiplication of complex numbers. Is it an integral domain? Is it a field? Justify your answer in each case. 4+1+1=6

(b) If R is the additive group of real numbers and R^+ the multiplicative group of positive real numbers, prove that the mapping $f : R \to R^+$ defined by $f(x) = e^x$ for all $x \in R$ is an isomorphism of R onto R^+. 5

(c) If R is a finite commutative ring with unity element, prove that every prime ideal of R is a maximum ideal of R. 4

UNIT—IV

7. *(a)* Prove that a commutative ring with unity, is a field if it has no proper ideals. 5

(b) If f is a homomorphism of a ring R into a ring R' with Kernel S, then prove that S is an ideal of R. 4

(c) Show that every ideal I of an integral domain R is of the form $I = Ra$ for some $a \in R$. 4

(d) Show that the polynomial $x^2 - 3$ is irreducible over the field of rational numbers. 2

8. *(a)* Define units, prime elements of a Euclidean ring and the unique factorization domain. 2+2+2=6

(b) Define the term 'associates' in a Euclidean domain. In $\mathbb{Z}_5$, are 2 and 3 associates? 3

(c) Prove that every prime element in an integral domain with unit element is irreducible. 3

(d) Prove that every field is a Euclidean ring. 3

UNIT—V

9. (a) Is $W = \{(x, 2x, 3x) : x \in \mathbb{R}\}$ a one-dimensional subspace of $\mathbb{R}^3$, where $\mathbb{R}$ is the field of real numbers? Justify your answer. 3

(b) If F is the field of real numbers, show that the vectors (1, 1, 0, 0), (0, 1, −1, 0), (0, 0, 0, 3) in $F^{(4)}$ are linearly independent over F. 3

(c) Determine whether or not the following vectors form a basis of R^3 :

(1, 1, 2), (1, 2, 5), (5, 3, 4) 3

(d) Prove that two finite dimensional vector spaces $V(F)$ and $U(F)$ over a field F are isomorphic if and only if $\dim U = \dim V$. 6

10. (a) Let T be a linear operator on R^3 defined by

$$T(x_1, x_2, x_3) = (3x_1 + x_3, -2x_1 + x_2, -x_1 + 2x_2 + 4x_3)$$

What is the matrix of T in the ordered basis $\{\alpha_1, \alpha_2, \alpha_3\}$, where $\alpha_1 = (1, 0, 1)$, $\alpha_2 = (-1, 2, 1)$ and $\alpha_3 = (2, 1, 1)$? 6

(b) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined as $T(a, b) = (a+b, a-b, b)$ be a linear transformation. Find the range, rank and nullity of T. 5

(c) Is the vector (2, −5, 3) in the subspace of $\mathbb{R}^3$ spanned by the vectors (1, −3, 2), (2, −4, −1), (1, −5, 7)? Justify your answer. 4

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2019

(October)

MATHEMATICS

(Honours)

(GHS-51)

(Elementary Number Theory and Advanced Algebra)

Marks : 75

Time : 3 hours

The figures in the margin indicate full marks for the questions

Answer **five** questions, choosing **one** from each Unit

Answer Elementary Number Theory and Advanced Algebra in two separate books.

(ELEMENTARY NUMBER THEORY)

UNIT—I

1. *(a)* State whether the following statements are True or False with brief justification (a, b, c, n denote integers) (any *five*) : 2×5=10

(i) If $a \mid bc$ and $\gcd(a, b) \neq 1$, then $a \mid c$.

(ii) If n is an odd integer, then $\gcd(n, n+2) = 1$.

(iii) If $a \mid bc$, then $a \mid b$ or $a \mid c$.

(iv) If a and b are odd integers, then $a^2 - b^2$ is divisible by 8.

(v) If

$$\gcd(a, p^2) = p \text{ and } \gcd(b, p^3) = p^2$$

where p is a prime, then

$$\gcd(ab, p^4) = p^2$$

(vi) If p is a prime and

$$p \mid (a^2 + b^2) \text{ and } p \mid (b^2 + c^2)$$

then $p \mid (a^2 - c^2)$.

(vii) If $a^3 \mid c^3$, then $a \mid c$.

(b) Find integers x and y which satisfy

$$\gcd(28, 72) = 28x + 72y$$ 3

(c) Exhibit ten consecutive integers such that all are composite integers. 2

2. *(a)* State and prove Euler's theorem. 1+5=6

(b) If p is an odd prime, then prove that

$$1^2 \cdot 3^2 \cdot 5^2 \cdots (p-2)^2 \equiv (-1)^{\frac{p+1}{2}} \pmod p$$ 4

(c) Show that the set

$$\{0, -7, 10, -5, 12, -3, 14, -1\}$$

is a complete residue system modulo 8. 3

(d) Prove that $n^2 \equiv 1 \pmod 8$ for any odd integer n. 2

UNIT—II

3. *(a)* State and prove Chinese remainder theorem. 1+5=6

(b) If m and n are relatively prime positive integers, then prove that

$$\phi(m \cdot n) = \phi(m) \cdot \phi(n)$$ 5

(c) Solve the linear congruence

$$6x \equiv 15 \pmod{21}$$ 4

4. *(a)* Let x and y be real numbers. Then prove that

$$[x] + [y] \le [x + y] \le [x] + [y] + 1$$ 4

(b) State and prove Mobius inversion formula. 1+4=5

(c) Compute $\sigma(n)$ and $\tau(n)$ for $n = 3600$. 2+2=4

(d) Prove that if $n = 2^k$ for some integer $k \ge 1$, then prove that $\phi(n) = \frac{n}{2}$. 2

(ADVANCED ALGEBRA)

UNIT—III

5. *(a)* Define normal subgroup. Give an example. 1+1=2

(b) Prove that a subgroup N of a group G is normal if

$$(xN)(yN) = (xy)N, \quad \forall x, y \in G$$ 3

(c) Prove that if K is a subgroup of a group G such that $g^2 \in K$, $\forall g \in G$, then K is normal in G. 3

(d) Prove that S_4 does not have a normal subgroup of order 3. 3

(e) Let G be a group, for $g \in G$, define $T_g : G \to G$ by $T_g(x) = gxg^{-1}$, $\forall x \in G$. Prove that T_g is an automorphism of G. 4

6. *(a)* Define integral domain. Prove that

$$\mathbb{Z}_n = \{0, 1, 2, \cdots, (n-1)\}$$

w.r.t. addition and multiplication modulo n is not an integral domain if n is not a prime. 1+3=4

(b) Define an ideal of a ring. Prove that

$$\mathbb{Z} = 98\,\mathbb{Z} + 99\,\mathbb{Z}$$ 1+2=3

(c) Let R be a ring with a unit element such that $a^2 = a$, $\forall a \in R$. Prove that—

(i) R is commutative;

(ii) every prime ideal of R is maximal. 4+4=8

UNIT—IV

7. *(a)* Prove that a finite integral domain is a field. 6

(b) Prove that any non-zero ring homomorphism from $\mathbb{Z}$ to $\mathbb{Z}$ is identity. 4

(c) What is a principal ideal domain (**PID**)? Is $\mathbb{Z}[x]$ a **PID**? Justify. 1+4=5

8. *(a)* Prove that if F is a field, then $F[x]$ is a Euclidean ring. 7

(b) Prove that

$$x^3 + x^2 + 1$$

is irreducible in $\mathbb{Z}_5[x]$. 2

(c) Prove that

$$\frac{\mathbb{Z}_5[x]}{\langle x^3 + x^2 + 1 \rangle}$$

is a field. 3

(d) How many elements are there in

$$\frac{\mathbb{Z}_5[x]}{\langle x^3 + x^2 + 1 \rangle}?$$

Justify. 3

UNIT—V

9. *(a)* Prove that a subset W of a vector space V over a field F is a subspace of V if and only if $\forall a, b \in F$ and $u, v \in W$, $au + bv \in W$. Hence show that the set

$$W = \{(x, y, z) \in \mathbb{R}^3 \mid 2x - 3y + z = 0\}$$

is a subspace of $\mathbb{R}^3$ over $\mathbb{R}$. 5+3=8

(b) If

$$U = \{(x, y) \in \mathbb{R}^2 \mid y = 2x\}$$
and $$W = \{(x, y) \in \mathbb{R}^2 \mid y = 3x\}$$

then show that $U \oplus W = V$ where $V = \mathbb{R}^2$, a vector space over the field $\mathbb{R}$. 4

(c) Let $B = \{v_1, v_2, \cdots, v_n\}$ be a basis of a finite dimensional vector space V over a field F. Show that every vector $v \in V$ can be expressed uniquely as a linear combination of vectors in B. 3

10. *(a)* Show that the mapping $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T(x, y, z) = (x + y, y + z, z + x)$$

is a linear transformation, where $\mathbb{R}^3$ is a vector space over $\mathbb{R}$. 3

(b) Let U and V be vector spaces over a field F. If $T : U \to V$ is a linear transformation, then show that—

(i) $\ker(T)$ is a subspace of U;

(ii) T is one-one if and only if $\ker(T) = \{0\}$, where $\ker(T)$ is the kernel of T. 2+4=6

(c) Consider the linear operator

$$T : \mathbb{R}^3 \to \mathbb{R}^3$$

defined by

$$T(x, y, z) = (2x, 4x - y, 2x + 3y - z)$$

Show that T is invertible and find T^{-1}. 3+3=6

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